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Some Relationships Between Dualities in String Theory

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ABSTRACT

Some relationships between string theories and eleven-dimensional supergravity are discussed and reviewed. We see how some relationships can be derived from others. The cases of $N = 2$ supersymmetry in nine dimensions and $N = 4$ supersymmetry in four dimensions are discussed in some detail. The latter case leads to consideration of quotients of a K3 surface times a torus and to a possible peculiar relationship between eleven-dimensional supergravity and the heterotic strings in ten dimensions.

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1 Introduction

Recent ideas concerning duality in string theory (such as [1, 2, 3]) have given hope to gaining insights into some non-perturbative form of string theory. Given the current status of string theory it is not easy to see how to prove such statements about duality. Rather, one can take the attitude that such dualities could be used, in part, as a defining property of string theory.

Given the many dualities that have been proposed, if we want to understand how to formulate a new form of string theory, it is important to know which dualities can be derived from the others. In particular we appear to have many forms of dualities relating theories with N supersymmetries in d -dimensional flat space-time for various values of N and d . In this talk I will give some simple ideas on how to formulate relationships between dualities by concentrating on the cases $N = 2, d = 9$ and $N = 4, d = 4$.

Much of this talk is not original and draws particularly heavily from [3] and the later sections are based on the collaborative work of [4]. Many aspects of section 3 were discussed in [5] although not in quite the same way as here. The way that the U -duality group is built up in section 5 is very similar in spirit to the work of [6]. It is hoped that the simple examples explained below show how the dualities can be directly related to each other in some contexts to build up a rather intricate picture.

In section 2 we will review the basic dualities used in this talk. In section 3 we do a “warm-up” exercise for the later sections. In section 4 we have an overview of four dimensional theories and in section 5 we look at the simplest example of an $N = 4$ theory in four dimensions.

2 Dualities

“Duality” is a much over-used word in the context we wish to use it and we need to refine our definitions somewhat. Firstly let us discuss U -duality as discussed in [2]. Consider a particular string theory. Such a theory will have some deformations (e.g., “truly marginal operators” in the language of conformal field theory) which will allow us to smoothly reach other string theories. Let us use \mathcal{M} to denote the moduli space of such theories. To avoid complications we will allow \mathcal{M} to include boundary points a finite distance away, but we will not allow ourselves to pass through the boundary to other theories by processes such as the one described in [7]. In simple cases one expects the moduli space to appear naturally in the form

$$\mathcal{M} = U \backslash \mathcal{T}, \tag{1}$$

where \mathcal{T} is some smooth domain and U is some discrete group acting upon it. \mathcal{T} is some generalized notion of a Teichmüller space and U is the group of U -dualities. We divide by discrete groups from the left as \mathcal{T} will typically be a right-coset as we will see later.

In general one can expect U to be generated by 3 subsets defined roughly as follows:

1. C -dualities: (This is not conventional notation.) These are equivalences in \mathcal{T} coming from the classical modular group. That is, if we can associate our string theory with some geometry, the classical moduli space of the geometric object will be $C \backslash \mathcal{T}$. The canonical example is $Sl(2, \mathbb{Z})$ for the moduli space of complex structures on a 2-torus.
2. T -dualities: These are further identifications within \mathcal{T} due to the conformal field theories associated with two different geometries being isomorphic. The canonical example is $R \leftrightarrow 1/R$ duality. In some conventions T is a group that contains C .
3. S -dualities: These are further identifications due to the effective quantum field theories associated to the string target space for two apparently different models being isomorphic. The canonical example is strong-weak string-coupling duality.

It is generally hoped that the full group U is generated completely by the elements of C , T and S .

Together with the notion of U -dualities we also have the concept of equivalences between theories which, at first sight, are qualitatively different. We list the ones needed in this talk below.

1. String-string duality. The type IIA superstring compactified on a K3 surface is equivalent to the heterotic string compactified on a 4-torus. We will denote this relationship by

$$(IIA \rightarrow \text{K3}) \cong (\text{Het} \rightarrow T^4). \quad (2)$$

This notion goes back as far as [8] but has been developed subsequently in many other references. The strongly coupled type II string corresponds to the weakly coupled heterotic string.

2. 11-dimensional supergravity as a string theory [9, 3].

$$(11d \rightarrow S^1) \cong \text{IIA}. \quad (3)$$

In this case the string coupling of the type II string becomes larger as the radius of the S^1 becomes larger.

3. Type II equivalences [10, 11].

$$(IIA \rightarrow S^1) \cong (\text{IIB} \rightarrow S^1). \quad (4)$$

In this case there is an $R \leftrightarrow 1/R$ relationship between the two S^1 s.

4. Heterotic equivalences [12, 13].

$$(\text{Het}_{E_8 \times E_8} \rightarrow S^1) \cong (\text{Het}_{SO(32)} \rightarrow S^1). \quad (5)$$

In this case the two 10-dimensional heterotic strings are different limits in the space $O(1, 17)/O(17)$.

It is fairly clear what is meant by each of the above equivalences with the exception of that for equation (3). Are we really meant to believe that eleven-dimensional supergravity on a circle is entirely equivalent to string theory? The answer to this question is probably no. In [3] this equivalence was more conservatively given as that between low-energy effective actions. We should be aware of this uncertainty whenever eleven-dimensional supergravity is mentioned below.

3 Nine Dimensions

We may now try to mix the ideas of U -duality and equivalences from the previous section. Consider the case of eleven-dimensional supergravity compactified on a 2-torus, T^2 . From the last section we therefore have

$$\begin{aligned} (11\text{d} \rightarrow T^2) &\cong [(11\text{d} \rightarrow S^1) \rightarrow S^1] \\ &\cong (\text{IIA} \rightarrow S^1) \\ &\cong (\text{IIB} \rightarrow S^1), \end{aligned} \tag{6}$$

thus relating eleven-dimensional supergravity to the IIB superstring.

Now let T^2 be given by a fundamental region in \mathbb{R}^2 in the usual way of the form of a rectangle with sides r_1 and r_2 . The starting point for our space of theories is thus a quadrant of \mathbb{R}^2 . Since the interchange of r_1 and r_2 clearly has no effect on the underlying theory, we should divide out by this interchange. This leads to an infinite triangle as shown in figure 1.

A generic point in this space corresponds to a nine-dimensional theory. When both radii go to infinity we obtain the eleven-dimensional theory. Consider the case when r_1 is finite and r_2 is infinite. This gives us the correspondence with the IIA theory as explained in [3]. Let us denote the IIA string coupling by $\lambda_A = \exp(\phi_A)$, where ϕ is the string dilaton. We then have

$$\lambda_A = r_1^{\frac{3}{2}}. \tag{7}$$

Thus the bottom left corner of figure 1 is the type IIA string in ten dimensions at zero string coupling.

An important point of [3] is that the ten space-time dimensions as seen by eleven-dimensional supergravity compactified on a circle are not quite the same ten space-time dimensions as seen by the type IIA superstring. They are related by a rescaling. This means that when r_2 is finite, it is not the radius of the circle on which the type IIA string is compactified. Denoting this latter radius by r_A we have

$$r_A = r_1^{\frac{1}{2}} r_2. \tag{8}$$

Now consider the type IIB interpretation. The effective field theories for the type IIA and IIB theories show that the respective dilatons must shift when the $R \leftrightarrow 1/R$ transformation

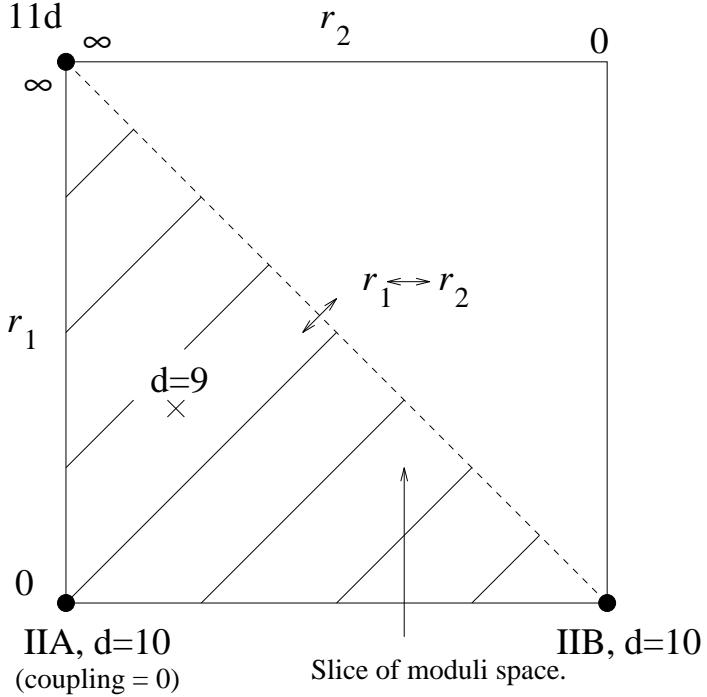


Figure 1: A slice of the space of theories in 9 dimensions.

of [10, 11] is performed. This means that we calculate the string coupling of the type IIB superstring as

$$\begin{aligned} \lambda_B &= \lambda_A \cdot r_A^{-1} \\ &= r_1 r_2^{-1}. \end{aligned} \tag{9}$$

Consider now the bottom right corner of figure 1. This now corresponds to the type IIB string in 10 dimensions but the coupling is not defined. We should really do a real blow-up at this point to get the correct moduli space. It is easy to see that the symmetry of the eleven-dimensional supergravity picture that exchanged the radii r_1 and r_2 now translates into the IIB superstring as

$$\lambda_B \leftrightarrow 1/\lambda_B. \tag{10}$$

That is, we have obtained S -duality for the IIB string.

Actually, we have not analyzed the complete moduli space. The moduli space of the torus should also allow the angle between the vectors of length r_1 and r_2 to vary. This gives the well-known result that the moduli space is actually the upper half plane divided by $Sl(2, \mathbb{Z})$. In the language of the type IIB superstring, this extra degree of freedom comes from the expectation value of the axion. Thus, the $Sl(2, \mathbb{Z})$ modular invariance of the torus on which eleven-dimensional supergravity was compactified can be used to “deduce” $Sl(2, \mathbb{Z})$ S -duality for the IIB string.

N	11d	II	Het
	Hol, X	Hol, X	Hol, X
8	$1, T^7$	$1, T^6$	—
4	$SU(2), K3 \times T^3$	$SU(2), K3 \times T^2$	$1, T^6$
2	$SU(3), CY \times S^1$	$SU(3), CY$	$SU(2), K3 \times T^2$
1	$G_2, Joyce$	—	$SU(3), CY$

Table 1: Four-dimensional theories obtained by compactification.

This S -duality for the type IIB string was conjectured in [2]. We see here that this conjecture is not independent of the others in the previous section.

4 Four Dimensional Theories

Let us consider obtaining four-dimensional theories by compactifying the known supersymmetric ten-dimensional string theories and eleven-dimensional supergravity over manifolds of six and seven dimensions respectively. The number of supersymmetries in four dimensions can be found by counting the number of covariantly constant spinors on the six and seven-dimensional manifolds. This in turn depends purely on the holonomy group of the compact manifold. In table 1 we list the number of supersymmetries in four dimensions for each higher dimensional theory.

This table requires some discussion. Firstly we only list possible geometric compactifications. By using more asymmetric methods, other models can be built such as an $N = 1$ theory built from the type II string (see, for example, [14]). Each of the entries in the table gives the holonomy of the compact space X , followed by an example of such a space where CY stands for a Calabi-Yau manifold. A “Joyce Manifold” is that of the type discovered in [15].

It is tempting to conjecture that for each of the rows in table 1 there is some equivalence between each of the entries. For the $N = 8$ this follows immediately from the conjectured equivalence between eleven-dimensional supergravity and the type IIA string by compactifying further on T^6 . Similarly the $N = 4$ row follows from equivalences mentioned earlier. Analysis on the $N = 2$ row was begun in [16, 17].

In some cases one can classify all the possibilities for X given the holonomy (see theorem 10.8 of [18]). For the rest of this section we will analyze the case of obtaining $N = 4$ supersymmetry from the type II string where this classification may be done. Any 6-dimensional manifold with holonomy $SU(2)$ must be of the form $(K3 \times T^2)/G$ where G acts freely. Any element $g \in G$ can be decomposed into an automorphism, g_1 , of the K3 surface and an auto-

morphism, g_2 , of the torus. To retain $SU(2)$ holonomy, these automorphisms must preserve the holomorphic 2-form and 1-form respectively. Such a g_1 necessarily has fixed points and so g_2 must act freely. Clearly then, if g is nontrivial, g_2 acts by a translation on the torus.

We can list all possibilities for the group G in this case. Since any nontrivial element of G must be fixed-point free, the associated g_2 must be nontrivial. Thus G is faithfully represented by translations in T^2 . It then follows that G must be of the form \mathbb{Z}_m or $\mathbb{Z}_m \times \mathbb{Z}_n$ for integers m, n . Without loss of generality we may assume that any element of G acts nontrivially on the K3 surface. From Nikulin's work [19] we can then list the possibilities for G . This is done in table 2. M is the rank of the maximal sublattice of $H^2(\text{K3}, \mathbb{Z})$ that transforms nontrivially under G .

Given a type II string compactified on a manifold $X_G = (\text{K3} \times T^2)/G$, it is natural to ask if this is equivalent to some orbifold of the heterotic string compactified on T^6 . That is, can we divide the $N = 4$ row in table 1 by G and maintain equivalences? This appears to be the case. From [20] we expect to identify the lattice of total cohomology $H^*(\text{K3}, \mathbb{Z})$ with the even-self dual lattice defining the heterotic string compactified on T^4 . Thus the action of G on $H^*(\text{K3}, \mathbb{Z})$ gives us a candidate for an asymmetric orbifold of the heterotic string. This appears to correspond precisely to the models studied in [21, 22]. This point has been investigated further in [23].

Two points are worth mentioning. Firstly, the asymmetric orbifolds of [22] should provide more examples of heterotic strings than we have listed here. This is because we have restricted our attention on the type II side to geometric quotients. Other models based on type II strings are possible, such as those of [24]. Secondly when one does an asymmetric orbifold it is important to check that the level-matching conditions of [25] are satisfied. Given the values of M in table 2 we can check whether this is so for all our examples. The answer is yes and, at least at first sight, this appears to be remarkable. This should be contrasted to cases where $N = 4$ supersymmetry is broken such as [17].

5 U -duality

In this section we will analyze the moduli space of the $N = 4$ theories as discussed in the last section. We will focus on the case of $\text{K3} \times T^2$ but it should be easy to extend this analysis to the quotients. A conjecture for the form of this moduli space was made in [2] by making some assumptions about the soliton spectrum. Here we will be able to rederive this result without making any direct reference to solitons but rather using the equivalences we already listed earlier. One can argue that these equivalences rest on details of the soliton spectrum

G	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	\mathbb{Z}_3	$\mathbb{Z}_3 \times \mathbb{Z}_3$	\mathbb{Z}_4	$\mathbb{Z}_4 \times \mathbb{Z}_4$	\mathbb{Z}_5	\mathbb{Z}_6	\mathbb{Z}_7	\mathbb{Z}_8
M	8	12	16	18	12	16	14	18	16	16	18	18

Table 2: Possible quotienting groups.

and so what we are doing in this section may be completely equivalent to [2]. Anyway the analysis below clearly shows the interrelation between such conjectures. This argument first appeared in [4] and the reader is referred there for details. The basic idea will be that we will take the moduli space of theories and try to identify a boundary. This process is not unique and the different boundaries will correspond to different interpretations of the theory. The mathematical principles of this process are in [26] but the reader is also referred to [3, 4] for a simpler treatment.

By general arguments from supergravity [27], the general form of the Teichmüller space for $N = 4$ theories in four dimensions is

$$\mathcal{T} \cong \frac{O(6, k)}{O(6) \times O(k)} \times \frac{Sl(2)}{U(1)}, \quad (11)$$

where k is the number of $N = 4$ matter supermultiplets. For ease of notation, when a coset is written $\frac{a}{b}$, the action is assumed to be from the right. In this case $k = 22$ (or $k = 22 - M$ for the cases listed in table 2). To form the moduli space we need to quotient by some group U .

From the conjecture concerning the rows of table 1, any point in $\mathcal{M} = U \backslash \mathcal{T}$ can be thought of as a compactification of eleven-dimensional supergravity, the type IIA or IIB superstring, or the two heterotic strings. Thus each point has five interpretations (compared to the three interpretations in figure 1). Given any one of these five interpretations we should be able to find part of the moduli space we already understand nonperturbatively.

Let us begin with the type IIB string. We will assume that we understand this theory in the weak-coupling limit, i.e., when $\lambda_B \rightarrow 0$. In this case we should just recover the Teichmüller space for conformal field theories with target space $K3 \times T^2$ [8] together with directions in the moduli space for deformations of the axion and 48 fields from the R-R sector. Thus we expect to be able to find a boundary of the form

$$\partial_{\lambda_B \rightarrow 0} \mathcal{T} \cong \frac{O(4, 20)}{O(4) \times O(20)} \times \frac{Sl(2)}{U(1)} \times \frac{Sl(2)}{U(1)} \times \mathbb{R}^{49}, \quad (12)$$

for the Teichmüller space. This is indeed the case following methods explained in [3, 4]. Now since we know that $O(4, 20; \mathbb{Z}) \times Sl(2, \mathbb{Z}) \times Sl(2, \mathbb{Z})$ acts on the above boundary, it must also be a subgroup of U .

We also know about another limit of this IIB theory. If we rescale the T^2 part so that its area becomes infinite then we are left with a type IIB string compactified on a K3 surface. The Teichmüller space we are left with should be that for IIB strings on a K3 surface [3] together with the complex structure and B -field for T^2 and the remaining R-R moduli. This is of the form

$$\partial_{T^2 \rightarrow \infty} \mathcal{T} \cong \frac{O(5, 21)}{O(5) \times O(21)} \times \frac{Sl(2)}{U(1)} \times \mathbb{R}^{26}, \quad (13)$$

which can also be found as a boundary. In this case we show that $O(5, 21; \mathbb{Z}) \times Sl(2, \mathbb{Z}) \subset U$. The relationship between the boundaries tells us the way these two subgroups fit together within U . Using methods such as those in [20] or [28] one can then show that

$$U \supseteq O(6, 22; \mathbb{Z}) \times Sl(2, \mathbb{Z}), \quad (14)$$

and that the equality must be satisfied if \mathcal{M} is Hausdorff [20].

No we have found U we can interpret \mathcal{M} in terms of the other strings. Firstly we can find the IIA string interpretation. Going to the weak-coupling string where we really understand what we are doing, this theory is mirror to the IIB theory. The action of the mirror map on (12) is within the $O(4, 20)$ factor but it exchanges the two $Sl(2)$ factors. Thus, the only noticeable effect is on the rôle of the $Sl(2, \mathbb{Z})$ factor in (14). For the IIB string this factor came from the complex structure of T^2 whereas now it acts as a T -duality on the radius and B -field as in [29].

Now consider the heterotic string. In this case we know that weakly coupled string has a Teichmüller space of $O(6, 22)/(O(6) \times O(22)) \times \mathbb{R}$ thanks to [12]. This is easy to fit into the required moduli space. In this case the only extra information coming from the U -duality group is the $Sl(2, \mathbb{Z})$ factor which forms the S -duality group.

Thus we see that for the type IIA, type IIB and heterotic string the rôle of the $Sl(2, \mathbb{Z})$ group is of a T , C and S duality respectively. This “triality of dualities” generalizes the work of [30] (and was independently investigated in [31] and recently in [32]).

Lastly we need to fit the interpretation of the four-dimensional model as a compactification of eleven-dimensional supergravity into the picture. Eleven-dimensional supergravity does not have any weak coupling limit since the coupling in the action is fixed. However, we can take the large radius limit. Actually, there are many large radius limits which appear to be qualitatively different.

In [3] it was argued that eleven-dimensional supergravity compactified on a K3 surface was equivalent to the heterotic string compactified on a 3-torus. Therefore, we should be able to identify eleven-dimensional supergravity compactified on $K3 \times T^3$. Sure enough, one of the boundaries of the space is

$$\partial_{R_1 \rightarrow \infty} \mathcal{T} \cong \frac{O(3, 19)}{O(3) \times O(19)} \times \frac{Sl(3)}{SO(3)} \times \frac{Sl(2)}{U(1)} \times \mathbb{R}^{69}, \quad (15)$$

where the first factor can be recognized as the Teichmüller space of Ricci-flat metrics on a K3 surface of fixed volume and the second factor as the Teichmüller space of flat metrics of fixed volume on T^3 . R_1 is some parameter such that the limit $R_1 \rightarrow \infty$ takes the volume of both the K3 surface and the 3-torus to infinity. The other degree of freedom for the two volumes is in the $Sl(2)/U(1)$ factor. It would thus appear that the $Sl(2, \mathbb{Z})$ factor in the U -duality group has yet another meaning for the eleven-dimensional supergravity picture.

It is interesting to note that as soon as we interpret our moduli space in terms of eleven-dimensional supergravity we recognize factors in the boundary which correspond *classical*

moduli spaces rather than the moduli spaces of conformal field theories we were seeing earlier. This is intimately connected with the fact that a conformal field theory moduli space has a classical moduli space on its boundary (from the $\alpha' \rightarrow 0$ limit).

Another boundary of the moduli space can be written as

$$\partial_{R_2 \rightarrow \infty} \mathcal{T} \cong \frac{O(2, 18)}{O(2) \times O(18)} \times \frac{Sl(4)}{SO(4)} \times \dots \quad (16)$$

The second factor is clearly the space of metrics on T^4 but what is the first factor? It can be written as part of the boundary of the space of Ricci-flat metrics on a K3 surface and so is the space of some kind of singular K3. One natural interpretation [4] is that the K3 surface has collapsed in itself and is now like a 3 dimensional object. We denote this as a “squashed K3”. Thus, our four-dimensional theory has an interpretation as eleven-dimensional supergravity compactified on a squashed K3 times T^4 (both being at large radius in the above limit).

Continuing this line of argument we can should be able to squash the K3 surface down to a 2-dimensional and then a 1-dimensional object. Actually there are two natural 1-dimensional limits, Ξ_1 and Ξ_2 depending on how we decompose the moduli space with respect to the lattice structure preserved by $O(4, 20; \mathbb{Z})$. This is tied to the fact that there are two even self-dual lattices in 16 dimensions. We now see that our four-dimensional theory can be thought of as eleven-dimensional supergravity compactified on $\Xi_i \times T^6$. We already knew however that it could also be thought of as the heterotic string compactified on T^6 .

The classical moduli space of the T^6 in the eleven-dimensional picture embeds nicely into the stringy moduli space of the T^6 in the heterotic picture and so it is tempting to “cancel” the two T^6 ’s against each other and make the bold assertion that the heterotic string in ten dimensions is equivalent to eleven-dimensional supergravity compactified on Ξ_i . Clearly the two choices of Ξ_i should give the $E_8 \times E_8$ string and the $SO(32)$ string. Whether this statement only makes sense in some delicate limit or whether we can really directly analyze compactification on Ξ_i remains unclear. In particular, we have not yet calculated what shape the Ξ_i ’s are. Clearly neither is a circle since we already know this should lead to the type IIA string. Thus, if they exist, they must be some more complicated 1-skeleton object. Clearly some degree of complexity is required from them since they contain the information about the gauge group of the heterotic string.

This prediction of some relation between eleven-dimensional supergravity and heterotic strings arises in this section in a way very similar to the way $Sl(2, \mathbb{Z})$ S -duality for the type IIB string arose in section 3. We see that often a full understanding of the moduli space of a given theory can tell us interesting things about the relationships between the associated higher-dimensional theories.

Note added

Since this talk was presented further constructions, in some ways similar to the orbifolds presented here and in [17], have been presented in [33, 34, 35, 36]. In general the $N < 4$ case appears to be more subtle than the version discussed above. Eleven-dimensional supergravity compactified on manifolds with G_2 holonomy has recently been discussed in [37, 38]. The recent paper [39] has some overlap with section 3.

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